1 Young agents

$$\max_{n,\ell,x\geq 0} \ln(wn-x)+\gamma\ln\left[\theta x^{\rho}+(1-\theta)\ell^{\rho}\right]^{1/\rho}+\lambda\ln(1-n-\ell)+(n+\ell)L \tag{1}$$

FOCs are:

$$-\frac{1}{wn-x}+\frac{\gamma\theta x^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}=0 \tag{2}$$

$$\frac{w}{wn-x}-\frac{\lambda}{1-\ell-n}+L=0 \tag{3}$$

$$\frac{\gamma(1-\theta)\ell^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}-\frac{\lambda}{1-\ell-n}+L=0 \tag{4}$$

Using (3) and (4),

$$\theta x^{\rho}+(1-\theta)\ell^{\rho}=\gamma(1-\theta)\ell^{\rho-1}\left(\frac{wn-x}{w}\right). \tag{5}$$

Using in (2), one can get

$$\left(\frac{\ell}{x}\right)^{\rho}=\left(\frac{\theta w}{1-\theta}\right)^{\rho/(\rho-1)}. \tag{6}$$

Note that, using the equation above, we can rewrite the term

$$\frac{\theta x^{\rho}+(1-\theta)\ell^{\rho}}{\gamma\theta x^{\rho-1}}=\frac{1}{\gamma}\left[x+\frac{1-\theta}{\theta}\left(\frac{\ell}{x}\right)^{\rho}x\right]=\frac{x}{\gamma}\left[1+\frac{1-\theta}{\theta}\left(\frac{\theta w}{1-\theta}\right)^{\rho/(\rho-1)}\right]\equiv x\phi_1. \tag{7}$$

Similarly,

$$\frac{\theta x^{\rho}+(1-\theta)\ell^{\rho}}{\gamma(1-\theta)l^{\rho-1}}=\frac{\ell}{\gamma}\left[1+\frac{\theta}{1-\theta}\left(\frac{1-\theta}{w\theta}\right)^{\rho/(\rho-1)}\right]\equiv\ell\phi_2. \tag{8}$$

Using (7) in (2),

$$x=\frac{wn}{1+\phi_1}. \tag{9}$$

Using (8) in (4),

$$1-\ell-n=\frac{\lambda\ell\phi_2}{1+L\ell\phi_2}. \tag{10}$$

Rearranging the equation above,

$$n=\frac{(1+L\ell\phi_2)(1-\ell)-\lambda\ell\phi_2}{1+L\ell\phi_2}. \tag{11}$$

Using (9), we can rewrite the term

$$wn - x = \left(\frac{\phi_1}{1+\phi_1}\right) wn. \tag{12}$$

Also, using (10), we can write

$$\frac{\lambda}{1-\ell-n} = \frac{1}{\ell\phi_2} + L. \tag{13}$$

Using (11), (12) and (13) in (3),

$$\left(\frac{1+\phi_1}{\phi_1}\right) \frac{1+L\ell\phi_2}{(1+L\ell\phi_2)(1-\ell) - \lambda\ell\phi_2} - \frac{1}{\ell\phi_2} = 0. \tag{14}$$

After a lot of algebra (see end of file),

$$\ell^2 L\phi_2(\phi_1+\phi_2+\phi_1\phi_2) + \ell(\phi_1+\phi_2+\phi_1\phi_2(1-L+\lambda)) - \phi_1 = 0. \tag{15}$$

If $L = 0$, there is a closed form solution for ℓ. If not, we just need to solve for the roots of a second degree polynomial. We should check if only one root is inside $(0, 1)$.

"A lot of algebra"

We can rewrite (14) as

$$(1+\phi_1)(1+L\ell\phi_2)\ell\phi_2 = \phi_1(1+L\ell\phi_2)(1-\ell) - \phi_1\lambda\ell\phi_2$$

$$\ell\phi_2 + L\ell^2\phi_2^2 + \phi_1\ell\phi_2 + \phi_1 L\ell^2\phi_2 = \phi_1 - \phi_1\ell + \phi_1\phi_2\ell L - \phi_1\phi_2 L\ell^2 - \phi_1\lambda\ell\phi_2$$

$$\ell^2\left(\phi_2^2 L + \phi_1 L\phi_2^2 + \phi_1\phi_2 L\right) + \ell\left(\phi_2 + \phi_1\phi_2 + \phi_1 - \phi_1\phi_2 L + \phi_1\phi_2\lambda\right) - \phi_1 = 0$$

We simplify

$$\phi_1 = \frac{1}{\gamma}\left[1 + \left(\frac{\theta}{1-\theta}\right)^{1/(\rho-1)} w^{\rho/(\rho-1)}\right]$$

$$\phi_2 = \frac{1}{\gamma}\left[1 + \left(\frac{1-\theta}{\theta}\right)^{1/(\rho-1)} w^{-\rho/(\rho-1)}\right]$$

2 Old agents

$$\max_{\ell, x \geq 0} \ln(\bar{w} - x) + \gamma \ln \left[\theta x^\rho + (1-\theta)\ell^\rho\right]^{1/\rho} + \lambda \ln(1-\ell) + \ell L \tag{16}$$

FOCs are:

$$-\frac{1}{\bar{w} - x} + \frac{\gamma \theta x^{\rho-1}}{\theta x^\rho + (1-\theta)\ell^\rho} = 0 \tag{17}$$

$$\frac{\gamma(1-\theta)\ell^{\rho-1}}{\theta x^\rho + (1-\theta)\ell^\rho} - \frac{\lambda}{1-\ell} + L = 0 \tag{18}$$

Manipulating (17),

$$\frac{1}{\bar{w} - x} = \frac{\gamma \theta x^{\rho-1}}{\theta x^\rho + (1-\theta)\ell^\rho} \iff \theta x^\rho + (1-\theta)\ell^\rho = (\bar{w} - x)\, \gamma \theta x^{\rho-1} \tag{19}$$

$$\iff \ell = \left[\frac{\theta}{1-\theta}\left(\bar{w}\gamma x^{\rho-1} - (1+\gamma)x^\rho\right)\right]^{1/\rho} \equiv \hat{\ell}(x). \tag{20}$$

That is, we can solve the problem of the old agent by guessing x, computing $\hat{\ell}(x)$ and verifying if (18) holds.

We need to be careful with the guesses that we make for x. First, x cannot be negative or greater than $\bar{w}$ (which would imply that consumption is negative). Second, x cannot be such that $\hat{\ell}(x)$ is negative or greater than one.

Note that we can rewrite $\hat{\ell}(x)$ as

$$\hat{\ell}(x) = \left[\frac{\theta}{1-\theta}\left(\bar{w}\gamma - (1+\gamma)x\right)\right]^{1/\rho} x^{(\rho-1)/\rho}. \tag{21}$$

Using (21), we can see that (remember that $\rho < 0$)

$$\lim_{x \to 0^+} \hat{\ell}(x) = 0. \tag{22}$$

Using (20), one can show that

$$\lim_{x \to \left(\frac{\gamma}{1+\gamma}\bar{w}\right)^-} \hat{\ell}(x) = \infty. \tag{23}$$

Also,

$$\frac{\partial \hat{\ell}(x)}{\partial x} = \frac{1}{\rho}\hat{\ell}(x)^{1-\rho}\frac{\theta}{1-\theta}\left(\bar{w}\gamma(\rho-1)x^{\rho-2} - (1+\gamma)\rho x^{\rho-1}\right) > 0 \tag{24}$$

$$\iff x < \bar{w}\frac{\gamma}{1+\gamma}\frac{\rho-1}{\rho}. \tag{25}$$

Note that

$$\frac{\gamma}{1+\gamma}\bar{w} < \bar{w}\frac{\gamma}{1+\gamma}\frac{\rho-1}{\rho}.$$

That is, for x between 0 and $\bar{w}\gamma/(1+\gamma)$, $\partial\hat{\ell}(x)/\partial x > 0$. This and equations (22) and (23) show that there is only one $\bar{x}$ between 0 and $\bar{w}\gamma/(1+\gamma)$ such that $\hat{\ell}(x) = 1$.

Therefore, in the code our guesses of x will be between 0 and $\bar{x}$.