$$H=\ln\left[w(n+\tau(v))-x\right]+\gamma\ln\left[\left(\theta x^{\rho}+(1-\theta)\ell^{\rho}\right)^{1/\rho}\right]+\lambda_d\ln(1-n-v-\ell)+\lambda_j\ln(1-n-\ell)+(n+\ell)L,$$

The term λ_j above also includes λ_p.

1 Case $n, v > 0$

$$\frac{\partial H}{\partial n}=\frac{w}{w(n+\tau(v))-x}-\frac{\lambda_d}{1-n-v-\ell}-\frac{\lambda_j}{1-n-\ell}+L=0 \tag{1}$$

$$\frac{\partial H}{\partial v}=\frac{w\tau'(v)}{w(n+\tau(v))-x}-\frac{\lambda_d}{1-n-v-\ell}=0 \tag{2}$$

$$\frac{\partial H}{\partial x}=-\frac{1}{w(n+\tau(v))-x}+\frac{\gamma\theta x^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}=0 \tag{3}$$

$$\frac{\partial H}{\partial \ell}=\frac{\gamma(1-\theta)\ell^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}-\frac{\lambda_d}{1-n-v-\ell}-\frac{\lambda_j}{1-n-\ell}+L=0 \tag{4}$$

Using (1) and (4),

$$\frac{w}{w(n+\tau(v))-x}=\frac{\gamma(1-\theta)\ell^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}.$$

Using (3),

$$\frac{1}{w(n+\tau(v))-x}=\frac{\gamma\theta x^{\rho-1}}{\theta x^{\rho}+(1-\theta)\ell^{\rho}}.$$

The two equations above imply that

$$\frac{x}{\ell}=\left(\frac{1-\theta}{w\theta}\right)^{1/(\rho-1)}\equiv\phi_3.$$

This implies that

$$\frac{\theta x^{\rho}+(1-\theta)\ell^{\rho}}{\gamma\theta x^{\rho-1}}=\frac{x}{\gamma}\left[1+\frac{1-\theta}{\theta}\left(\frac{\ell}{x}\right)^{\rho}\right]\equiv x\phi_1,$$

where ϕ_1 only depends on parameters.
Using this in (3),

$$x=\frac{w(n+\tau(v))}{1+\phi_1}, \tag{5}$$

and

$$\ell=\frac{\phi_3^{-1}w(n+\tau(v))}{1+\phi_1}. \tag{6}$$

We can write

$$w(n+\tau(v)) - x = w(n+\tau v) - \frac{w(n+\tau(v))}{1+\phi_1} = \frac{\phi_1}{1+\phi_1} w(n+\tau(v)). \tag{7}$$

Also,

$$1-n-v-\ell = \frac{(1-n-v)(1+\phi_1) - \phi_3^{-1} w(n+\tau(v))}{1+\phi_1}.$$

Using these in (2),

$$\frac{w\tau'(v)(1+\phi_1)}{\phi_1 w(n+\tau(v))} = \frac{\lambda_d(1+\phi_1)}{(1-n-v)(1+\phi_1) - \phi_3^{-1} w(n+\tau(v))}.$$

Solving for n,

$$n = \frac{(1-v)(1+\phi_1)\tau'(v) - \tau(v)\tau'(v)\phi_3^{-1} w - \tau(v)\lambda_d\phi_1}{\phi_1\lambda_d + \tau'(v)(1+\phi_1) + \tau'(v)\phi_3^{-1} w}.$$

Using

$$\tau(v) = \alpha_1 v - \alpha_2 v^2, \qquad \tau'(v) = \alpha_1 - 2\alpha_2 v,$$

we have that

$$n = \frac{N_n^3(v)}{D_n^1(v)},$$

where $N_n^3(v)$ is a third-degree polynomial of v, and $D_n^1(v)$ is a first-degree polynomial of v (N stands for numerator and D stands for denominator). Also,

$$n + \tau(v) = \frac{N_n^3(v)}{D_n^1(v)} + \tau(v) = \frac{N_n^3(v) + \tau(v) D_n^1(v)}{D_n^1(v)} \equiv \frac{N_{n+\tau(v)}^3(v)}{D_n^1(v)}. \tag{8}$$

Using (6),

$$\begin{gathered} \ell = \frac{\phi_3^{-1} w}{1+\phi_1}(n+\tau(v)) = \frac{\phi_3^{-1} w N_{n+\tau(v)}^3(v)}{(1+\phi_1) D_n^1(v)} \equiv \frac{N_\ell^3(v)}{D_\ell^1(v)}. \\ 1-v-n-\ell = 1 - v - \frac{N_n^3(v)}{D_n^1(v)} - \frac{N_\ell^3(v)}{(1+\phi_1) D_n^1(v)} = \\ \frac{(1-v)(1+\phi_1) D_n^1(v) - (1+\phi_1) N_n^3(v) - N_\ell^3(v)}{(1+\phi_1) D_n^1(v)} \equiv \frac{N_{1-v-n-\ell}^3(v)}{D_{1-v-n-\ell}^1(v)}. \end{gathered} \tag{9}$$

Similarly,

$$1-n-\ell = \frac{(1+\phi_1) D_n^1(v) - (1+\phi_1) N_n^3(v) - N_\ell^3(v)}{(1+\phi_1) D_n^1(v)} \equiv \frac{N_{1-n-\ell}^3(v)}{D_{1-n-\ell}^1(v)}. \tag{10}$$

Using (7) in (1),

$$\frac{(1+\phi_1)w}{\phi_1 w(n+\tau(v))} - \frac{\lambda_d}{1-n-v-\ell} - \frac{\lambda_j}{1-n-\ell} + L = 0.$$

Using (8), (9), and (10) in the equation above,

$$\frac{(1+\phi_1)D^1_n(v)}{\phi_1 N^3_{n+\tau(v)}(v)} - \frac{\lambda_d D^1_{1-v-n-\ell}(v)}{N^3_{1-v-n-\ell}(v)} - \frac{\lambda_j D^1_{1-n-\ell}(v)}{N^3_{1-n-\ell}(v)} + L = 0.$$

We can write this equation as

$$\frac{A_0 + A_1 v}{B_0 + B_1 v + B_2 v^2 + B_3 v^3} + \frac{C_0 + C_1 v}{D_0 + D_1 v + D_2 v^2 + D_3 v^3} + \\ \frac{E_0 + E_1 v}{F_0 + F_1 v + F_2 v^2 + F_3 v^3} + L = 0.$$

Multiplying both sides by the product of the three denominators, we get an equation saying that a 9th-degree polynomial must be equal to zero.